Time Dilation

The time required for an event to be completed is called the “elapsed time.”

Different observers of the event will measure correct elapsed times, depending on their motion relative to the event:

*Observers moving relative to an event measure a longer elapsed time than do observers stationary relative to the event.*

--Albert Einstein, 1905

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<th>Elapsed times measured by observers at rest relative to the event measure the “proper time.”</th>
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<td>( T_0 = “\text{proper time}” )</td>
<td>Suppose ( v/c = 0.80: )</td>
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<td>Elapsed times measured by observers moving relative to the event measure the “dilated time.”</td>
<td>( 1 - (v/c)^2 = 1 - 0.80^2 )</td>
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<td>( T = “\text{dilated time}” )</td>
<td>( = 1 - 0.64 )</td>
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<td>( T = \frac{T_0}{\sqrt{1 - (\frac{v}{c})^2}} )</td>
<td>( = 0.36 )</td>
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<td>Denominator is always less than or equal to 1.0, so ( T &gt; T_0. )</td>
<td>( \sqrt{1 - (v/c)^2} = \sqrt{0.36} )</td>
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<td>( = 0.60 )</td>
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<td>( T = T_0 / 0.60 )</td>
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<td>( = 1.67 \ T_0 )</td>
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<td>The dilated time is 67% longer than the proper time.</td>
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Verb: *to dilate*, lengthen, widen
Example A:

An observer stationary relative to the melting of the ice at the right says it took eight minutes for the ice to melt.

Proper Time: \( T_0 = 8.0 \) minutes

How long does an observer, moving at a speed of four-tenths the speed of light say it took for the ice to melt?

\[
T = T_0 \left[ \left(1 - \frac{v}{c}\right)^2 \right]^{1/2} = \frac{8.00}{\left(1 - 0.40^2\right)^{1/2}} = 8.73 \text{ minutes}
\]

Dilated Time: 8.73 minutes

Example B: A runner completes a journey around the track in 2.00 minutes, according to her. Does she measure the dilated time, or the proper time?

Answer:

Like the observer who was stationary relative to the melting of the ice, this observer is stationary relative to the running of the runner (herself). Therefore, the runner measures the proper elapsed time.

\( T_0 = 2.00 \) minutes
Length Contraction

The lengths of objects moving relative to an observer are contracted.

\[ L = L_0 \sqrt{1 - \left(\frac{v}{c}\right)^2} \]

\( L_0 \) = Length seen by an observer at rest relative to the object = “the proper length”

\( L \) = Length seen by an observer moving relative to the object = “the contracted length”

Contraction occurs in the dimension that is along the direction of motion. Below we show an object moving at various speeds to the right, along the x-axis. The narrowing of the object’s width occurs along the object’s x-axis.
Example:

The distance from Earth to a certain star is $6.0 \times 10^{18}$ m, as measured by an Earth observer. He measures the proper length because he is at rest relative to the path connecting Earth to the star.

$L_o = 6.0 \times 10^{18}$ m

(a) What would a space-traveler leaving Earth and headed toward the star at speed $v = 2.4 \times 10^8$ m/s say is the distance from Earth to the star?

Note: $v/c = 0.80$

The space-traveler is moving relative to the path, so he measures the contracted path length:

$L = L_o \left[1 - (v/c)^2\right]^{1/2}$

$= 6.0 \times 10^{18} \left(1 - 0.80^2\right)^{1/2}$

$= 3.6 \times 10^{18}$ m

(b) How long does the space-traveler say it took to get to the star?

Time = Distance / Speed

$= 3.6 \times 10^{18}$ m / $2.4 \times 10^8$ m/s

$= 1.5 \times 10^{10}$ s

The traveler is stationary relative to the traveling of the traveler (himself), just like the runner was stationary relative to the running of the runner, who measured the proper running time.

Thus, the traveler in this current example measures the proper time $T_o$ for the journey to the star.

$T_o = 1.5 \times 10^{10}$ s

(c) What is the time measured by the Earth observer?

Calculate the time using two different methods.

1. Time = Distance / Speed

$T = \frac{6.0 \times 10^{18}}{2.4 \times 10^8}$

$= 2.5 \times 10^{10}$ s

2. Using the result of Part (b):

$T = \frac{1.5 \times 10^{10}}{(1 - 0.80^2)^{1/2}}$

$= 2.5 \times 10^{10}$ s

Previous results reveal a general truth: Travelers (runners, hikers, walkers) measure the proper time it takes to complete the journey.
Example: An observer, stationary with respect to a certain path, measures the proper length of the path. She says the length is 400 m:

\[ L_0 = 400 \text{ m} \]

A second person--a runner--runs along that path at a speed of 0.70 \( c \):

\[ v = 0.70 \times (3.0 \times 10^8 \text{ m/s}) = 2.1 \times 10^8 \text{ m/s} \]

(a) What does the runner say is the length of the path?

The runner is moving relative to the path, so he measures the contracted length, \( L \):

\[
L = L_0 \left[ 1 - \left( \frac{v}{c} \right)^2 \right]^{1/2} \\
= 400 \left( 1 - 0.70^2 \right)^{1/2} \\
= 286 \text{ m}
\]

(b) How much time does the runner say it took to complete the run, traveling at 0.90 \( c \)?

Time = Distance / Speed
\[ = \frac{286 \text{ m}}{2.1 \times 10^8 \text{ m/s}} \]
\[ = 1.36 \times 10^{-6} \text{ s} \]

(c) What does the stationary observer say is the time it took to complete the journey?

Calculate this time using two different methods.

1. \[ T = \frac{\text{Distance}}{\text{Speed}} = \frac{400 \text{ m}}{2.1 \times 10^8 \text{ m/s}} = 1.90 \times 10^{-6} \text{ s} \]

2. \[ T = \frac{T_o}{\left[ 1 - \left( \frac{v}{c} \right)^2 \right]^{1/2}} = \frac{1.36 \times 10^{-6}}{\left( 1 - 0.70^2 \right)^{1/2}} = 1.90 \times 10^{-6} \text{ s} \]
Example:

The catcher on a baseball team says the path connecting home plate to first base has a length of 27.40 m. He is at rest relative to the path, so he measures the proper length, $L_o$.

$L_o = 27.40 \text{ m}$

The catcher observes a runner traveling from home plate to first base at a speed of 0.90 c.

$v = 0.90 (3.0 \times 10^8) \text{ m/s}
= 2.7 \times 10^8 \text{ m/s}$

(a) What does the catcher say is the time it takes the runner to reach first base?

$\text{Time} = 27.40 \text{ m} / 2.7 \times 10^8 \text{ m/s}
= 1.01 \times 10^{-7} \text{ s}$

The catcher is moving relative to the event (the runner), so the time calculated above is the dilated time:

$T = 1.01 \times 10^{-7} \text{ s}$

(b) What does the runner say is the time it takes to reach first base?

She measures the proper time:

$T_o = T \left[1 - (v/c)^2\right]^{1/2}
= 1.01 \times 10^{-7} \left(1 - 0.90^2\right)^{1/2}
= 4.40 \times 10^{-8} \text{ s}$

(c) What does the runner say is the length of the path?

The runner is moving relative to the path, so she measures the contracted length:

$L = L_o \left[1 - (v/c)^2\right]^{1/2}
= 27.40 \left(1 - 0.90^2\right)^{1/2}
= 11.94 \text{ m}$

(d) What does the runner say her speed is?

$\text{Speed} = \text{Distance} / \text{Time}
= (11.94 \text{ m}) / (4.40 \times 10^{-8} \text{ s})
= 2.7 \times 10^8 \text{ m/s}$

Note that this is the same speed the catcher measures.

The two persons disagree about distance between home plate and first base, and they disagree about the time it takes the runner to run from home plate and first base, but they agree about the runner’s speed.
In previous examples involving a traveling observer and a stationary observer, we assumed that each observer measured the same speed for the traveling observer.

We prove below that this is true in general.

To simplify our equations, let $\beta$ represent the radical $\sqrt{1 - (v/c)^2}$

**Traveling Observer Measures:**

Distance = $L$
Time = $T_o$
Speed = $L/T_o$

**Stationary Observer Measures:**

Distance = $L_o$
Time = $T$
Speed = $L_o/T$

The ratio of the speeds:

$$(L/T_o) \div (L_o/T) = (L/L_o) \cdot (T/T_o)$$

$= \beta \cdot (1/\beta)$

$= 1$$
Energy Transformed into Mass and Mass into Energy

The initial total energy of an object of mass $m$, at rest is called the “rest-mass energy,” and is given by

$$m_0 c^2$$

If the object absorbs or releases a quantity $Q$ of heat or electromagnetic energy (EM), the new total energy is

$$mc^2 = m_0 c^2 + Q$$

$$(m - m_0)c^2 = Q$$

$$\Delta m = \frac{Q}{c^2}$$

If $Q > 0$, energy entered the object.

If $Q < 0$, energy is released.

Example:

600 J of light energy is absorbed by a 10-kg object. What is the new mass of the object?

$$\Delta m = \frac{Q}{c^2}$$

$$= 600 / (3.0 \times 10^8)^2$$

$$= 5.0 \times 10^{-14} \text{ kg}$$

$$m = 10,000,000,000,000,000,005 \text{ kg}$$
Example:

Light energy from the sun arrives on Earth at an average daytime rate of about 1400 W/m$^2$. The surface area of a sun-bather is about 1.2 m$^2$.

(a) What will be the increase in mass of the sun-bather owing to the energy she receives in four hours via sunlight, assuming that half of the energy is absorbed, while the remainder is reflected away?

4.0 hours = 14,400 seconds

\[ Q = \left(\frac{1}{2}\right) (1400 \text{ W/m}^2) (1.2 \text{ m}^2) (14,400 \text{ s}) \]
\[ = 1.21 \times 10^7 \text{ J} \]

\[ \Delta m = \frac{Q}{c^2} \]
\[ = \frac{(1.21 \times 10^7 + 0)}{(3.0 \times 10^8)^2} \]
\[ = 1.34 \times 10^{-10} \text{ kg} \]

(b) How much mass would be gained if the sunbather remained in the sunlight for 100 years?

Answer: Her mass would increase by 0.03 grams.
Pair Production

Under the right conditions, a pulse of electromagnetic energy (a “photon”) can be completely transformed into an electron and a positron. What is the minimum energy (in MeV) a photon can have in order that it be converted into an electron-positron pair?

Electron Rest-mass: 9.11 \times 10^{-31} \text{ kg}

Positron Rest-mass: 9.11 \times 10^{-31} \text{ kg}

Total Change in mass = 2 \times 9.11 \times 10^{-31} \text{ kg}

\[ \Delta m = 18.22 \times 10^{-31} \text{ kg} \]

\[ Q/c^2 = \Delta m \]
\[ Q = c^2 (\Delta m) \]
\[ Q = (3.0 \times 10^8)^2 (18.22 \times 10^{-31}) \]
\[ = 1.64 \times 10^{13} \text{ J} \]
\[ = 1.64 \times 10^{13} \text{ J}/1.6 \times 10^{-19} \text{ J/eV} \]
\[ = 1.02 \times 10^6 \text{ eV} \]
\[ = 1.02 \text{ MeV} \]

If the gamma-ray photon doesn’t have at least this much energy, the pair will not be created.
Fusion

The most important example of a conversion of mass into heat and electromagnetic energy occurs in the interior of the sun where “deuterium” ($^1\text{H}_2$) and “tritium” ($^1\text{H}_3$) isotopes of hydrogen combine (“fuse”) to form helium.

The mass of the helium nucleus is less than the mass of the hydrogen isotopes by the amount $3 \times 10^{-29}$ kg:

$$\Delta m = -3 \times 10^{-29} \text{ kg}$$

$$Q/c^2 = \Delta m$$

$$Q = (3 \times 10^8)^2 (-3 \times 10^{-29})$$

$$= -2.70 \times 10^{-12} \text{ J}$$

Energy Released = $2.7 \times 10^{-12}$ J (heat and electromagnetic radiation)
Total Energy of a Moving Object

If the object is not at rest, the total energy is given by the equation below:

\[ E = mc^2 / \left[ 1 - \left( \frac{v}{c} \right)^2 \right]^{1/2} \]

Note that when the object is at rest, \( v = 0 \), and \( E = mc^2 \).

Relativistic Kinetic Energy

vs

Classical Kinetic Energy

The kinetic energy of a moving object is the difference between its total energy and the rest-mass energy. Take away the rest portion of the energy from the total energy, and what’s left is the motion part of the energy—the kinetic energy:

\[ K = E - mc^2 \]
\[ = mc^2 /\left[1 - \left( \frac{v}{c} \right)^2 \right]^{1/2} - mc^2 \]

Note that if \( v = 0 \), then \( K = 0 \), as expected for an object not moving.

The kinetic energy above is called the relativistic kinetic energy. It is the exact kinetic energy. The equation below is used to calculate the approximate kinetic energy: it’s called the classical kinetic energy.

\[ K = \frac{1}{2} mv^2 \]
Example:

(a) What is its rest-mass energy of an electron (in MeV)?

Note: 1.0 electron-volt (eV) = 1.6 x 10^{-19} J 
1 MeV = 1 x 10^6 eV

\[ E = mc^2 \]
\[ = 9.11 \times 10^{-31} (3.0 \times 10^8)^2 \]
\[ = 8.20 \times 10^{-14} J \]
\[ = 8.20 \times 10^{-14} J / 1.6 \times 10^{-19} J/eV \]
\[ = 0.51 \times 10^6 eV \]
\[ = 0.51 MeV \]

(b) What is the total energy of an electron moving at 0.9995 c?

\[ E = mc^2 / \left[ 1 - (v/c)^2 \right]^{1/2} \]
\[ = 0.51 / (1 - 0.9995^2)^{1/2} \]
\[ = 16.13 MeV \]

(c) What is the kinetic energy?

\[ K = E - mc^2 \]
\[ = 16.13 - 0.51 \]
\[ = 15.62 MeV \]

(d) What is the kinetic energy using the old equation—the “classical” equation?

\[ K = \frac{1}{2} mv^2 \]
\[ = \frac{1}{2} (9.11 \times 10^{-31})(0.9995 \times 3.0 \times 10^8)^2 \]
\[ = 4.10 \times 10^{-14} J \]
\[ = 4.10 \times 10^{-14} J / 1.6 \times 10^{-19} J/eV \]
\[ = 0.26 \times 10^6 eV \]
\[ = 0.26 MeV \]

At very high speeds, the classical kinetic energy equation is wildly inaccurate, as we see above, but at lower speeds the classical kinetic energy equation is extremely accurate, as we will see on the next page.
Prove that the classical kinetic energy is virtually the same as the relativistic kinetic energy.

By the “binomial expansion,”

\[(1 - x)^{1/2} = 1 + \frac{1}{2} x, \ x \ll 1\]

Let \(x = (v/c)^2\)

If \(v \ll c\):

\[
K = mc^2 /\sqrt{1 - x^{1/2}} - mc^2 = mc^2 (1 - x)^{1/2} - mc^2 = mc^2 (1 + \frac{1}{2} x) - mc^2 = \frac{1}{2} mc^2 x = \frac{1}{2} mc^2 (v/c)^2 = \frac{1}{2} mv^2
\]

Example:

Calculate and compare the classical and relativistic kinetic energies (in joules) of a proton traveling at the “non-relativistic” speed \(v/c = 0.05, \ (v = 1.5 \times 10^7 \text{ m/s})\).

Classical:

\[
K = \frac{1}{2} (1.67 \times 10^{-27}) (1.5 \times 10^7)^2 = 1.8788 \times 10^{-13} \text{ J}
\]

Relativistic:

\[
mc^2 = 1.67 \times 10^{-27} (3.0 \times 10^8)^2 = 1.503 \times 10^{-10} \text{ J}
\]

\[
K = mc^2 /\sqrt{1 - (v/c)^2}^{1/2} - mc^2 = 1.503 \times 10^{-10} / (1 - 0.05^2)^{1/2} - 1.503 \times 10^{-10} = 1.8823 \times 10^{-13} \text{ J}
\]

The classical kinetic energy is 99.8% of the relativistic kinetic energy.
Example:

How much work (in MeV) would need to be done on a proton to increase its speed from 0.60c to 0.70c?

Mass of proton: \( m = 1.67 \times 10^{-27} \text{ kg} \)

\[
mc^2 = 1.67 \times 10^{-27} (3.0 \times 10^8)^2 \\
= 1.50 \times 10^{-10} \text{ J} \\
= 1.50 \times 10^{-10} \text{ J} / 1.6 \times 10^{-19} \text{ J/eV} \\
= 9.38 \times 10^8 \text{ eV} \\
= 938 \times 10^6 \\
= 938 \text{ MeV}
\]

\[
K = mc^2 / [1 - (v/c)^2]^{1/2} - mc^2 \\
= 938 / (1 - 0.70^2)^{1/2} - 938 \\
= 375.5 \text{ MeV}
\]

\[
K_0 = mc^2 / [1 - (v_0/c)^2]^{1/2} - mc^2 \\
= 938 / (1 - 0.60^2)^{1/2} - 938 \\
= 234.5 \text{ MeV}
\]

Use the work-kinetic energy theorem:

\[
W = K - K_0 \\
= 375.5 - 234.5 \\
= 141 \text{ MeV}
\]