According to Albert Einstein’s Special Theory of Relativity, mass can be converted to energy and energy converted to mass:

\[ \Delta m = \frac{Q}{c^2} \]

\[ Q = \Delta m \cdot c^2 \]

**Example A:**

600 J of light energy Q is absorbed by a 10-kg object. What is the new mass of the object?

\[ \Delta m = \frac{Q}{c^2} \]
\[ = \frac{600}{(3.0 \times 10^8)^2} \]
\[ = 6.67 \times 10^{-15} \text{ kg} \]

\[ m = 10,000,000,000,000,067 \text{ kg} \]

**Example B:**

Light energy from the sun arrives on Earth at an average daytime rate of about 1400 W/m².

The surface area of a 60-kg sun-bather is about 1.2 m².

What will be the gain in the mass of the sun-bather owing to the energy she receives in four hours via sunlight, assuming all of the energy is absorbed?

Absorbed Power = 1400 W/m² (1.2 m² )
\[ = 1680 \text{ W} \]

4.0 hours = 14,400 seconds

Joules Absorbed = Joules/Second x Num Secs
\[ = (1680 \text{ J/s}) \times 14,400 \text{ s} \]
\[ = 2.42 \times 10^7 \text{ J} \]

\[ \Delta m = \frac{Q}{c^2} \]
\[ = \frac{2.42 \times 10^7}{(3.0 \times 10^8)^2} \]
\[ = 2.69 \times 10^{-10} \text{ kg} \]

**Example C:**

A 10-kg object is lifted 100 meters to the top of a building.

By how much does its mass change?

\[ Q = mgh \]
\[ = 10 (9.8) (100) \]
\[ = 9800 \text{ J} \]

\[ \Delta m = \frac{Q}{c^2} \]
\[ = \frac{9800}{(3.0 \times 10^8)^2} \]
\[ = 1.09 \times 10^{-13} \text{ kg} \]
Fusion

The most important example of a conversion of mass into energy--such as heat and electromagnetic energy--occurs in the interior of the sun where two different isotopes of hydrogen, “deuterium” (\(^1\)H\(^2\)) and “tritium” (\(^1\)H\(^3\)), combine (“fuse”) to form helium.

The mass of the helium nucleus is less than the mass of the hydrogen isotopes by the amount \(3 \times 10^{-29}\) kg:

\[ \Delta m = -3 \times 10^{-29}\text{ kg} \]

\[ Q = (\Delta m) c^2 = (-3 \times 10^{-29})(3 \times 10^8)^2 \]

\[ = -2.70 \times 10^{12}\text{ J} \]

Energy Released per Fusion = \(2.7 \times 10^{12}\) J

Example:

Each second, the Sun releases an estimated \(3.846 \times 10^{26}\) joules of energy in the form of light and other forms of radiation.

How much mass is lost second?

\[ \Delta m = Q/c^2 = (-3.846 \times 10^{26})/9 \times 10^{16} \]

\[ = -4.27 \times 10^9\text{ kg} \]

More than four billion kilograms of mass is lost by the Sun each second, and in its places appears an equivalent amount of energy.
Electron-Positron Annihilation

**BEFORE**

The electron’s antiparticle is the positron. If they encounter each other, they will annihilate each other and in their place appears two gamma rays whose total energy is the sum of the two particle’s rest-mass energies.

The electron and positron in this example are assumed to be at rest, and therefore the total momentum before annihilation is zero.

**AFTER**

Photon energies are identical, in order that momentum be conserved.
Total Energy

According to Einstein, the total energy of an object moving with speed $v$ is:

$$E = mc^2 / (1 - \beta^2)^{1/2}$$

$\beta = v/c$, where $c = 3.00 \times 10^8$ m/s.

Note that when the object is at rest, $\beta = 0$, and $E = mc^2$, which is called the “rest-mass energy.”

Relativistic Kinetic Energy

The kinetic energy of a moving object is the difference between its total energy and the rest-mass energy.

Take away the rest portion ($mc^2$) of the energy from the total energy, then what’s left is the motion part of the energy--the correct kinetic energy--the “relativistic kinetic energy.”

$$K = E - mc^2$$
$$= mc^2 / (1 - \beta^2)^{1/2} - mc^2$$

Note that if $v = 0$, then $K = 0$, as expected for an object not moving.
Other Energy Units

An electron placed next to the negative plate in the figure accelerates to the positive plate. Without proof we state that its kinetic energy when it arrives at the positive plate is $1.6 \times 10^{-19}$ J.

$1.0$ electron-volt (eV) = $1.6 \times 10^{-19}$ J

$1.0$ million eV = $1.0$ MeV
Example:

Note: electron mass = 9.11 x 10^{-31} kg

(a) What is the rest-mass energy of an electron (in MeV)?

\[ E = mc^2 \]
\[ = 9.11 \times 10^{-31} \times (3.0 \times 10^8)^2 \]
\[ = 8.20 \times 10^{-14} \text{ J} \]
\[ = 8.20 \times 10^{-14} \text{ J} / 1.6 \times 10^{-19} \text{ J/eV} \]
\[ = 0.51 \times 10^6 \text{ eV} \]
\[ = 0.51 \text{ MeV} \]

(b) What is the total energy of an electron moving at 0.9995 c?

\[ \beta = 0.9995 \]
\[ E = mc^2 / (1 - \beta^2)^{1/2} \]
\[ = 0.51 / (1 - 0.9995^2)^{1/2} \]
\[ = 16.13 \text{ MeV} \]

(c) What is its relativistic kinetic energy?

\[ K = E - mc^2 \]
\[ = 16.13 - 0.51 \]
\[ = 15.62 \text{ MeV} \]

(d) What is the kinetic energy using the “classical” equation?

\[ K = \frac{1}{2} m v^2 \]
\[ = \frac{1}{2} (9.11 \times 10^{-31})(0.9995 \times 3.0 \times 10^8)^2 \]
\[ = 4.10 \times 10^{-14} \text{ J} \]
\[ = 4.10 \times 10^{-14} \text{ J} / 1.6 \times 10^{-19} \text{ J/eV} \]
\[ = 0.26 \times 10^6 \text{ eV} \]
\[ = 0.26 \text{ MeV} \]

At very high speeds, the classical kinetic energy equation is wildly inaccurate, as we see above, but at lower speeds the classical
kinetic energy equation is extremely accurate, as we will see on the next page.

Example:

Calculate and compare the classical and relativistic kinetic energies (in joules) of a proton traveling at the “non-relativistic” speed, $v = 0.01c$ ($\beta = 0.01$).

\[ v = 0.01 \times (3.0 \times 10^8) \]
\[ = 3.0 \times 10^6 \text{ m/s} \]

Classical: \[ K = \frac{1}{2} mv^2 \]
\[ = \frac{1}{2} (1.67 \times 10^{-27}) (3.0 \times 10^6)^2 \]
\[ = 7.515 \times 10^{-15} \text{ J} \]

Relativistic:

\[ mc^2 = 1.67 \times 10^{-27} (3.0 \times 10^8)^2 \]
\[ = 1.503 \times 10^{-10} \text{ J} \]

\[ K = \frac{mc^2}{(1 - \beta^2)^{1/2}} - mc^2 \]
\[ = 1.503 \times 10^{-10} (1 - 0.01^2)^{1/2} - 1.503 \times 10^{-10} \]
\[ = 7.516 \times 10^{-15} \text{ J} \]

Even at speeds as large as the current one--3.0 million m/s, the classical kinetic energy is still accurate to within one-hundredth of one percent.
Example:

How much work (in MeV) would need to be done on a proton to increase its speed from $\beta_0 = 0.60$ to $\beta = 0.70$?

Mass of proton: $m = 1.67 \times 10^{-27}$ kg

\[
mc^2 = 1.67 \times 10^{-27} \times (3.0 \times 10^8)^2 = 1.50 \times 10^{-10} \text{ J} \\
= 1.50 \times 10^{-10} \text{ J} / 1.6 \times 10^{-19} \text{ J/eV} \\
= 9.39 \times 10^8 \text{ eV} \\
= 939 \times 10^6 \\
= 939 \text{ MeV}
\]

\[K = mc^2 / (1 - \beta^2)^{1/2} - mc^2 \]
\[= 939 / (1 - 0.70^2)^{1/2} - 939 \\
= 375.9 \text{ MeV}\]

\[K_0 = mc^2 / (1 - \beta_0^2)^{1/2} - mc^2 \]
\[= 939 / (1 - 0.60^2)^{1/2} - 939 \\
= 234.8 \text{ MeV}\]

Use the work-kinetic energy theorem:

\[W = K - K_0 \]
\[= 375.9 - 234.8 \\
= 141.1 \text{ MeV}\]
Time Dilation

The time required for an event to occur depends on the observer’s motion relative to the event.

*Observers moving relative to an event measure a longer time than do observers stationary relative to the event.*

--Albert Einstein, 1905

Times measured by observers at rest relative to the event are called the “proper time.”

\[ T_o = \text{“proper time”} \]

Times measured by observers moving relative to the event are called the “dilated time.”

\[ T = \text{“dilated time”} \]
\[ = T_o / (1 - \beta^2)^{1/2} \]

For moving objects, the denominator is always less than 1.0, so \( T > T_o \).

Note: the word “proper” in the words “proper time” does not imply that the time measured by this observer is the only “real” time, and time dilated measurements are somehow flawed because of errors by the observer, or faulty equipment, or an illusion. The dilated time measured by the observer moving relative to the event is just as real for that observer as it is for the observer who is at rest relative to the event.
Time dilation is not a visual thing; it is not an illusion; it is a real lengthening of time.

**Example A:**

An observer at rest relative to an ice cube says it took eight minutes for the ice to melt.

Proper Time: $T_o = 8.0$ minutes

How long does an observer, moving at a speed of four-tenths the speed of light say it took for the ice to melt?

$\beta = 0.40$

$T = T_o / (1 - \beta^2)^{1/2}$

$= 8.00 / (1 - 0.40^2)^{1/2}$

$= 8.73$ minutes

Dilated Time: 8.73 minutes

**Example B:**

A runner completes a journey around the track in 2.00 minutes, according to her. Does she measure the dilated time, or the proper time?

The runner is located wherever the running is occurring, so she measures the proper time.

$T_o = 2.00$ minutes

Suppose a passenger in a spaceship flying overhead at 0.80 c observes the runner. How long will he say it took the runner to complete the journey around the track?

$T = 2.00 / (1 - 0.80^2)^{1/2}$

$= 3.33$ minutes
Example:

A hovering drone in a hypothetical universe observes an automobile drive at speed \( v = 40 \text{ m/s} \) across town in 14 minutes. The driver says the trip took only 12 minutes; what is the speed of light in that universe?

The event is the \textit{moving} of the automobile, and it is occurring wherever the automobile is at any moment. The driver is inside the automobile, so she is at rest relative the location where the event is occurring; therefore, she measures the proper time.

The drone’s location relative to the place where the moving of the automobile is occurring is constantly changing. The drone therefore is moving relative to the event, so the it measures the dilated time.

\[ T_0 = 12 \text{ minutes} \]
\[ T = 14 \text{ minutes} \]

Assume Einstein’s Special Theory holds true in the hypothetical universe.

\[ T = T_0 / (1 - \beta^2)^{1/2} \]
\[ 14 = 12 / (1 - \beta^2)^{1/2} \]
\[ \beta = 0.515 \]
\[ 40/c = 0.515 \]
\[ c = 77.67 \text{ m/s} \]
Muons

Muons are unstable particles similar to electrons, but much heavier. They are created when protons streaming toward Earth from collide with oxygen and nitrogen nuclei in the upper atmosphere.

Muons after creation travel at $\beta = 0.9993$, and decay via emission of massless particles called neutrinos, and leave behind electrons in their place.

According to an observer moving with a muon, the muon on average decays after $2.2 \times 10^{-6}$ s. This is the *proper* decay time. A *classical* distance calculations shows that an observer moving with a muon created in the upper atmosphere would travel only a few hundred meters:

$$\text{Distance} = \text{Speed} \times \text{Time}$$
$$= (0.9993)(3.0 \times 10^8)(2.2 \times 10^{-6})$$
$$= 659 \text{ m}$$

However, according to Earth observers, muons in fact travel much farther before decaying. The explanation for this difference lies in the fact that the Earth observer measures the dilated time:

$$T = T_o / (1 - \beta^2)^{1/2}$$
$$= (2.2 \times 10^{-6}) / (1 - 0.9993^2)^{1/2}$$
$$= 58.8 \times 10^{-6} \text{ s}$$

This time is about 27 times the proper time, so the Earth observer says the muon travels 27 times farther than the muon says—about 18,000 meters, and both are right.
Length Contraction

\[ L = L_0 \left(1 - \beta^2\right)^{1/2} \]

\( L_0 \) = Length seen by an observer at rest relative to the object
   = “the proper length”

\( L \) = Length seen by an observer moving relative to the object
   = “the contracted length”

Both lengths are correct for their respective observers.

Contraction occurs in the dimension that is along the direction of motion. Below we show an object moving at various speeds to the right, along the x-axis. The narrowing of the object’s width occurs along the object’s x-axis.

Note: the word “proper” in the words “proper length” does not imply that the length measured by this observer is the only “real” length, and all other measurement are somehow flawed because of errors by the observer, faulty equipment, or an illusion. The contracted length measured by the observer moving relative to the object is just as real for that observer as it is for the observer who is at rest relative to the object.

Length contraction is not a visual thing; it is not an illusion; it is a real contraction.
Example:

The distance from Earth to a certain star is $6.0 \times 10^{18}$ m long, as measured by an Earth observer. He measures the *proper* length of the path because he is at rest relative to it.

$L_0 = 6.0 \times 10^{18}$ m

(a) What would a space-traveler, leaving Earth and headed toward the star at $2.4 \times 10^8$ m/s ($\beta = 0.80$), say is the distance from Earth to the star?

The space-traveler is *moving* relative to the path, so he measures the *contracted* length of the line segment:

$L = L_0 (1 - \beta^2)^{1/2}
= 6.0 \times 10^{18} (1 - 0.80^2)^{1/2}
= 3.6 \times 10^{18}$ m

(b) How long does the space-traveler say it took to get to the star?

The event in question is the traveling of the space traveler; the event is occurring wherever the traveler is at any instant, so the traveler is stationary relative to the event. Therefore, the traveler measures the proper time, as calculated below:

Time = Distance Measured by Traveler/ Speed
$T_0 = 3.6 \times 10^{18}$ m/ $2.4 \times 10^8$ m/s
$= 1.5 \times 10^{10}$ s
$= 1.5 \times 10^{10}$ s.

(c) What is the time measured by the Earth observer?

The Earth observer is moving relative to the event, so he measures the dilated time.

Time = Distance Measured by Earth Observer / Speed
$T = 6.0 \times 10^{18}/2.4 \times 10^8
= 2.5 \times 10^{10}$ s

(d) Calculate the dilated time using the time-dilation equation.

$T = 1.5 \times 10^{10} / (1 - 0.80^2)^{1/2}
= 2.5 \times 10^{10}$ s
Example:

An observer standing next to a certain path measures its length to be 400 m. The observer is at rest relative to the path, so he measures the proper length:

\[ L_0 = 400 \text{ m} \]

A person runs along that path at a speed of 0.70 c:

\[ v = 0.70 \times (3.0 \times 10^8 \text{ m/s}) = 2.1 \times 10^8 \text{ m/s} \]

(a) What does the runner say is the length of the path?

The runner is moving relative to the path, so he measures the contracted length:

\[ L = L_0 \left(1 - \beta^2\right)^{1/2} \]
\[ = 400 \times (1 - 0.70^2)^{1/2} \]
\[ = 286 \text{ m} \]

(b) How much time does the runner say it took to complete the run?

Time = Distance / Speed
\[ = 286 \text{ m} / 2.1 \times 10^8 \text{ m/s} \]
\[ = 1.36 \times 10^{-6} \text{ s} \]

The event in question is the running; the runner is located wherever the running is happening. The runner therefore is at rest relative to the event; the runner measures the proper time:

\[ T_0 = 1.36 \times 10^{-6} \text{ s} \]

(c) What does the stationary observer say is the time it took to complete the journey?

Calculate the journey time using two different methods.

1. Time = Distance / Speed
\[ = 400 \text{ m} / 2.1 \times 10^8 \text{ m/s} \]
\[ = 1.90 \times 10^{-6} \text{ s} \]

2. \[ T = T_0 / (1 - \beta^2)^{1/2} \]
\[ = 1.36 \times 10^{-6} / (1 - 0.70^2)^{1/2} \]
= 1.90 \times 10^{-6} \text{ s}
Example:

The catcher on a baseball team says the path connecting home plate to first base has a length of 27.40 m. He is at rest relative to the path, so he measures the proper length, \( L_o \).

\[ L_o = 27.40 \text{ m} \]

The catcher observes a runner traveling from home plate to first base at a speed of 0.90 c.

\[ v = 0.90 \cdot (3.0 \times 10^8) \]
\[ = 2.7 \times 10^8 \text{ m/s} \]

(a) What does the catcher say is the time it takes the runner to reach first base?

Time = Distance/Speed:
\[ = \frac{27.40 \text{ m}}{2.7 \times 10^8 \text{ m/s}} \]
\[ = 1.01 \times 10^{-7} \text{ s} \]

The event, which is the running, is taking place at the runner’s location. The catcher is moving relative to the event’s location (the runner’s location), so the catcher measures the dilated time:

\[ T = 1.01 \times 10^{-7} \text{ s} \]

(b) What does the runner say is the time it takes to reach first base?

The runner measures the proper time:

\[ T = T_o / (1 - \beta^2)^{1/2} \]
\[ 1.01 \times 10^{-7} = T_o / (1 - 0.90^2)^{1/2} \]
\[ T_o = 4.40 \times 10^{-8} \text{ s} \]

(c) What does the runner say is the length of the path?

The runner is moving relative to the path, so he measures the contracted length:

\[ L = L_o (1 - \beta^2)^{1/2} \]
\[ = 27.40 (1 - 0.90^2)^{1/2} \]
\[ = 11.94 \text{ m} \]

(d) What does the runner say her speed is?

Runner Measures Distance = 11.94 m
Runner Measures Time = 4.40 \times 10^{-8} \text{ s}

Speed = Distance / Time
\[ = (11.94 \text{ m}) / (4.40 \times 10^{-8} \text{ s}) \]
\[ = 2.7 \times 10^8 \text{ m/s} \]

Note that this is the same speed the catcher measures. The two persons disagree about distance between home plate and first base, and they disagree about the time it takes the runner to run from home plate and first base, but they agree about the runner’s speed, i.e., they agree about the ratio of distance and time. The catcher says the distance is longer, and the time is longer; the runner says the distance is shorter, and the time is shorter, but the ratios of distance over times—the speeds—are the same.
Classical Relative Velocities

Classical relative velocities are velocities that are negligibly small compared to the speed of light.

Let the velocity of an Object A relative to an Observer O be called $V_{AO}$.

Let the velocity of Object B relative to Observer O be called $V_{BO}$.

Objects moving to the left have negative velocities, while objects moving to the right have positive velocities.

The velocity of A relative to B is

$$V_{AB} = V_{AO} - V_{BO}$$

The velocity of B relative to A is the opposite:

$$V_{BA} = - V_{AB}$$

Classical Relative Speed

Speeds are absolute values of velocities, so the relative speed, which we will symbolize as $v$, is equal to either one of the expressions below:

$$v = |V_{AB}|$$

or

$$v = |V_{BA}|$$
Example:

(a) What is the velocity of A relative to B?

\[ V_{AB} = V_{AO} - V_{BO} = 20 \text{ m/s} - (-30 \text{ m/s}) = 50 \text{ m/s} \]

(b) What is the velocity of B relative to A?

\[ V_{BA} = -V_{AB} = -50 \text{ m/s} \]

(c) What is the relative speed between the two objects?

\[ v = |V_{AB}| = 50 \text{ m/s} \]

or

\[ v = |V_{BA}| = 50 \text{ m/s} \]
(a) What is the velocity of A relative to B?

\[ V_{AB} = V_{AO} - V_{BO} \]
\[ = -40 - (-50) \]
\[ = 10 \text{ m/s} \]

(b) What is the velocity of B relative to A?

\[ V_{BA} = -V_{AB} \]
\[ = -10 \text{ m/s} \]

(c) What is the relative speed between the two objects?

\[ v = | -10 \text{ m/s} | \text{ or } v = | 10 \text{ m/s} | \]
\[ \beta \beta v = 10 \text{ m/s} \]

Relativistic Relative Velocities

At very great velocities--greater than about one-hundredth the speed of light, Newtonian physics, i.e., non-relativistic physics, is no longer reliable.

The equation that is valid for relativistic, as well as classical, velocities is shown below for objects A and B moving relative to an observer O:

\[ V_{AB} = (V_{AO} - V_{BO}) / (1 - V_{AO}V_{BO}/c^2) \]

Note that if \( V_{BO} \) and \( V_{AO} \) are negligibly small compared to \( c \),

\[ V_{AO}V_{BO}/c^2 \approx 0 \]

and therefore \( V_{AB} = V_{AO} - V_{BO} \), which is the classical result.
Relativistic Relative *Speeds*

Speeds are absolute values of velocities, so the relativistic relative speed, which we will symbolize as \( v \), is equal to either one of the two expressions below:

\[ v = |V_{AB}| \]

or

\[ v = |V_{BA}| \]

Either choice gives the same answer.
Example:

Spaceship A is moving at a velocity 0.30 c relative to observers on Earth, as it chases Spaceship B, which is moving relative to Earth at 0.50 c. Spaceship B is emitting light at a frequency $f_s = 6.00 \times 10^{15}$ Hz. What frequency do observers in Spaceship A see?

Note that Spaceship B is pulling away from Spaceship A, so a red shifting of the light occurs, so we expect to see below that the observed frequency is “redder,” i.e., lower.

First, we will determine the relative speed between the two spaceships. Recall that speeds are absolute values of velocities, so the relative speed $v$ we’re looking for is the absolute value of the relative velocity—either $|V_{AB}|$, or $|V_{BA}|$.

$$V_{AB} = \frac{(V_{AO} - V_{BO})}{1 - \frac{V_{AO}V_{BO}}{c^2}}$$

$$= \frac{(0.30 \text{ c} - 0.50 \text{ c})}{1 - 0.50 \times 0.30}$$

$$= 0.24 \text{ c}$$

and

$$V_{BA} = -0.24 \text{ c}$$

Relative speed:

$$v = |V_{AB}|$$

$$= |0.24 \text{ c}|$$

$$= 0.24 \text{ c}.$$}

Recall the Doppler Effect for Light equation discussed in Chapter 24, which uses the relative speed, v:

$$f_o = f_s (1 \pm \frac{v}{c})$$

Recession is occurring, so we use the negative sign to ensure that the observed frequency is less than the broadcast frequency:

$$f_o = 6.00 \times 10^{15} (1 - 0.24)$$

$$= 4.56 \times 10^{15} \text{ Hz}$$