According to Albert Einstein’s Special Theory of Relativity, mass can be converted to energy, and energy converted to mass:

\[ \Delta m = \frac{Q}{c^2} \]

\[ Q = (\Delta m) c^2 \]

**Example A:**

600 J of light energy \( Q \) is absorbed by a 10-kg object. What is the new mass of the object?

\[ \Delta m = \frac{Q}{c^2} \]
\[ = \frac{600}{(3.0 \times 10^8)^2} \]
\[ = 5.0 \times 10^{-14} \text{ kg} \]

\[ m = 10.000 \ 000 \ 000 \ 000 \ 000 \ 005 \text{ kg} \]

**Example B:**

Light energy from the sun arrives on Earth at an average daytime rate of about 1400 W/m².

The surface area of a 60-kg sun-bather is about 1.2 m².

What will be the gain in the mass of the sun-bather owing to the energy she receives in four hours via sunlight, assuming all of the energy is absorbed?

4.0 hours = 14,400 seconds

\[ Q = (1400 \text{ (J/s) } /\text{m}^2) \times (1.2 \text{ m}^2) \times (14,400 \text{ s}) \]
\[ = 2.42 \times 10^7 \text{ J} \]

\[ \Delta m = \frac{Q}{c^2} \]
\[ = \frac{(2.42 \times 10^7)/(3.0 \times 10^8)^2}{\text{kg}} \]
\[ = 2.69 \times 10^{-10} \text{ kg} \]
Fusion

The most important example of a conversion of mass into heat and electromagnetic energy occurs in the interior of the sun where “deuterium” (\(^1\text{H}^2\)) and “tritium” (\(^1\text{H}^3\)) combine (“fuse”) to form helium.

The mass of the helium nucleus is less than the mass of the hydrogen isotopes by the amount \(3 \times 10^{-29}\) kg:

\[
\Delta m = -3 \times 10^{-29} \text{ kg}
\]

\[
Q = (\Delta m) c^2
\]
\[
= (-3 \times 10^{-29}) (3 \times 10^8)^2
\]
\[
= -2.70 \times 10^{-12} \text{ J}
\]

Energy Released per Fusion = \(2.7 \times 10^{-12}\) J

Example:

Each second, the Sun releases an estimated \(3.846 \times 10^{26}\) joules of energy in the form of light and other forms of radiation. How many fusions take place each second?

\[
(3.846 \times 10^{26} \text{ J/s}) / 2.7 \times 10^{-12} \text{ J} = 1.4 \times 10^{38} \text{ per second}
\]
Total Energy

According to Einstein, if an object is moving with speed $v$, its total energy is:

$$E = mc^2 / (1 - \beta^2)^{1/2}$$

where $\beta = v/c$ and $c = 3.00 \times 10^8$ m/s

Note that when the object is at rest, $\beta = 0$, and $E = mc^2$, which is called the “rest-mass energy.”

Relativistic Kinetic Energy

The kinetic energy of a moving object is the difference between its total energy and the rest-mass energy.

Take away the rest portion ($mc^2$) of the energy from the total energy, then what’s left is the *motion* part of the energy--the correct kinetic energy--the “relativistic kinetic energy.”

$$K = E - mc^2$$
$$= mc^2 / (1 - \beta^2)^{1/2} - mc^2$$

Note that if $v = 0$, then $K = 0$, as expected for an object not moving.
Other Energy Units

An electron placed next to the negative plate in the figure accelerates to the positive plate. Without proof we state that its kinetic energy when it arrives at the positive plate is $1.6 \times 10^{-19}$ J.

$1.0$ electron-volt (eV) = $1.6 \times 10^{-19}$ J

$1.0$ million eV = $1.0$ MeV
Example:

Note: electron mass = 9.11 x 10^{-31} kg

(a) What is the rest-mass energy of an electron (in MeV)?

\[ E = mc^2 \]
\[ = 9.11 \times 10^{-31} \times (3.0 \times 10^8)^2 \]
\[ = 8.20 \times 10^{-14} \text{ J} \]
\[ = 8.20 \times 10^{-14} \text{ J} / 1.6 \times 10^{-19} \text{ J/eV} \]
\[ = 0.51 \times 10^6 \text{ eV} \]
\[ = 0.51 \text{ MeV} \]

(b) What the total energy of an electron moving at 0.9995 c?

\[ \beta = 0.9995 \]
\[ E = mc^2 / (1 - \beta^2)^{1/2} \]
\[ = 0.51 / (1 - 0.9995^2)^{1/2} \]
\[ = 16.13 \text{ MeV} \]

(c) What is its relativistic kinetic energy?

\[ K = E - mc^2 \]
\[ = 16.13 - 0.51 \]
\[ = 15.62 \text{ MeV} \]

(d) What is the kinetic energy using the old (incorrect) equation--the “classical” equation?

\[ K = \frac{1}{2} mv^2 \]
\[ = \frac{1}{2} (9.11 \times 10^{-31})(0.9995 \times 3.0 \times 10^8)^2 \]
\[ = 4.10 \times 10^{-14} \text{ J} \]
\[ = 4.10 \times 10^{-14} \text{ J} / 1.6 \times 10^{-19} \text{ J/eV} \]
\[ = 0.26 \times 10^6 \text{ eV} \]
\[ = 0.26 \text{ MeV} \]

At very high speeds, the classical kinetic energy equation is wildly inaccurate, as we see above, but at lower speeds the classical kinetic energy equation is extremely accurate, as we will see on the next page.
Example:

Calculate and compare the classical and relativistic kinetic energies (in joules) of a proton traveling at the “non-relativistic” speed, $\beta = 0.01$.

$v/c = 0.01$
$v = 3.0 \times 10^6 \text{ m/s}$

Classical: $K = \frac{1}{2}mv^2$

$$= \frac{1}{2} (1.67 \times 10^{-27}) (3.0 \times 10^6)^2$$

$$= 7.515 \times 10^{-15} \text{ J}$$

Relativistic:

$mc^2 = 1.67 \times 10^{-27} (3.0 \times 10^8)^2$

$$= 1.503 \times 10^{-10} \text{ J}$$

$$K = mc^2 / (1 - \beta^2)^{1/2} - mc^2$$

$$= 1.503 \times 10^{-10} (1 - 0.01^2)^{1/2} - 1.503 \times 10^{-10}$$

$$= 7.516 \times 10^{-15} \text{ J}$$

Even at speeds as large as the current one--3.0 million m/s, the classical kinetic energy is still accurate to within one-hundredth of one percent.
Example:

How much work (in MeV) would need to be done on a proton to increase its speed from $\beta_0 = 0.60$ to $\beta = 0.70$?

Mass of proton: $m = 1.67 \times 10^{-27}$ kg

\[
mc^2 = 1.67 \times 10^{-27} (3.0 \times 10^8)^2 \\
= 1.50 \times 10^{-10} \text{ J} \\
= 1.50 \times 10^{-10} \text{ J} / 1.6 \times 10^{-19} \text{ J/eV} \\
= 9.38 \times 10^8 \text{ eV} \\
= 938 \times 10^6 \\
= 938 \text{ MeV}
\]

\[
K = mc^2 / (1 - \beta^2) - mc^2 \\
= 938 / (1 - 0.70^2) - 938 \\
= 375.5 \text{ MeV}
\]

\[
K_0 = mc^2 / (1 - \beta_0^2) - mc^2 \\
= 938 (1 - 0.60^2)^{-1/2} - 938 \\
= 234.5 \text{ MeV}
\]

Use the work-kinetic energy theorem:

\[
W = K - K_0 \\
= 375.5 - 234.5 \\
= 141 \text{ MeV}
\]
Time Dilation

The time required for an event to occur depends on the observer’s motion relative to the event.

*Observers moving relative to an event measure a longer time than do observers stationary relative to the event.*

--Albert Einstein, 1905

Times measured by observers at rest relative to the event are called the “proper time.”

\[ T_o = \text{“proper time”} \]

Times measured by observers moving relative to the event are called the “dilated time.”

\[ T = \text{“dilated time”} \]
\[ = T_o / (1 - \beta^2)^{1/2} \]

For moving objects, the denominator is always less than 1.0, so \( T > T_o \).

Note: The word “proper” does not imply that the time is the “true” time and therefore that the dilated time is untrue. Both times are correct for the respective observers.
Example A:

An observer at rest relative to an ice cube says it took eight minutes for the ice to melt.

Proper Time: $T_0 = 8.0$ minutes

How long does an observer, moving at a speed of four-tenths the speed of light say it took for the ice to melt?

$\beta = 0.40$

$T = T_0 / (1 - \beta^2)^{1/2}$

$= 8.00 / (1 - 0.40^2)^{1/2}$

$= 8.73$ minutes

Dilated Time: 8.73 minutes

Example B:

A runner completes a journey around the track in 2.00 minutes, according to her. Does she measure the dilated time, or the proper time?

The runner is located wherever the running is occurring, so she measures the proper time.

$T_0 = 2.00$ minutes

Suppose a passenger in a spaceship flying overhead at 0.80 c observes the runner. How long will he say it took the runner to complete the journey around the track?

$T = 2.00 / (1 - 0.80^2)^{1/2}$

$= 3.33$ minutes
Example:

A hovering drone in a hypothetical universe observes an automobile drive at speed \( v = 40 \text{ m/s} \) across town in 14 minutes. The driver says the trip took only 12 minutes; what is the speed of light in that universe?

The event is the *moving* of the automobile, and it is occurring wherever the automobile is at any moment. The driver is inside the automobile, so she is at rest relative the location where the event is occurring; therefore, she measures the proper time.

The drone’s location relative to the place where the moving of the automobile is occurring is constantly changing. The drone therefore is moving relative to the event, so the it measures the dilated time.

\[ T_0 = 12 \text{ minutes} \]
\[ T = 14 \text{ minutes} \]

Assume Einstein’s Special Theory holds true in the hypothetical universe.

\[
T = T_0 / \left( 1 - \beta^2 \right)^{1/2}
\]
\[
14 = 12 / \left( 1 - \beta^2 \right)^{1/2}
\]
\[
\beta = 0.515
\]
\[
40/c = 0.515
\]
\[
c = 77.67 \text{ m/s}
\]
Length Contraction

\[ L = L_0 (1 - \beta^2)^{1/2} \]

\( L_0 \) = Length seen by an observer *at rest* relative to the object
   = “the proper length”

\( L \) = Length seen by an observer *moving* relative to the object
   = “the contracted length”

Both lengths are correct.

Contraction occurs in the dimension that is along the direction of motion. Below we show an object moving at various speeds to the right, along the x-axis. The narrowing of the object’s width occurs along the object’s x-axis.
Example:

The path stretching from Earth to a certain star is $6.0 \times 10^{18}$ m long, as measured by an Earth observer. He measures the *proper* length of the path because he is at rest relative to it.

$L_0 = 6.0 \times 10^{18}$ m

(a) What would a space-traveler, leaving Earth and headed toward the star at eight-tenths the speed of light ($\beta = 0.80$), say is the distance from Earth to the star?

The space-traveler is *moving* relative to the path, so he measures the *contracted* length of the line segment:

$L = L_0 \left(1 - \beta^2\right)^{1/2}
= 6.0 \times 10^{18} \left(1 - 0.80^2\right)^{1/2}
= 3.6 \times 10^{18}$ m

(b) How long does the space-traveler say it took to get to the star?

The event in question is the traveling of the space traveler; the event is occurring wherever the traveler is at any instant, so the traveler is stationary relative to the event. Therefore, the traveler measures the proper time.

\[ v = 2.4 \times 10^8 \text{ m/s} \]

Time = Distance Measured by Traveler / Speed
\[ T_0 = \frac{3.6 \times 10^{18} \text{ m}}{2.4 \times 10^8 \text{ m/s}}
= 1.5 \times 10^{10} \text{ s} \]

= 1.5 x 10^{10} s.

(c) What is the time measured by the Earth observer?

The Earth observer is moving relative to the event, so he measures the dilated time.

Time = Distance Measured by Earth Observer / Speed
\[ T = \frac{6.0 \times 10^{18}}{2.4 \times 10^8}
= 2.5 \times 10^{10}$ s

(d) Calculate the dilated time using the time-dilation equation.

\[ T = \frac{1.5 \times 10^{10}}{(1 - 0.80^2)^{1/2}}
= 2.5 \times 10^{10}$ s
Example:

An observer standing next to a certain path measures its length to be 400 m. The observer is at rest relative to the path, so he measures the proper length:

\[ L_0 = 400 \text{ m} \]

A person runs along that path at a speed of 0.70 c:

\[ v = 0.70 \times (3.0 \times 10^8 \text{ m/s}) = 2.1 \times 10^8 \text{ m/s} \]

(a) What does the runner say is the length of the path?

The runner is moving relative to the path, so he measures the contracted length:

\[ L = L_0 \left(1 - \beta^2\right)^{1/2} \]
\[ = 400 \left(1 - 0.70^2\right)^{1/2} \]
\[ = 286 \text{ m} \]

(b) How much time does the runner say it took to complete the run?

Time = Distance / Speed
\[ = \frac{286 \text{ m}}{2.1 \times 10^8 \text{ m/s}} \]
\[ = 1.36 \times 10^{-6} \text{ s} \]

The event in question is the running; the runner is located wherever the running is happening. The runner is at rest relative to the event, and therefore he measures the proper time.

\[ T_0 = 1.36 \times 10^{-6} \text{ s} \]

(c) What does the stationary observer say is the time it took to complete the journey?

Calculate the journey time using two different methods.

1. Time = Distance / Speed
\[ = \frac{400 \text{ m}}{2.1 \times 10^8 \text{ m/s}} \]
\[ = 1.90 \times 10^{-6} \text{ s} \]

2. \[ T = \frac{T_0}{(1 - \beta^2)^{1/2}} \]
\[ = \frac{1.36 \times 10^{-6}}{(1 - 0.70^2)^{1/2}} \]
\[ = 1.90 \times 10^{-6} \text{ s} \]
Example:

The catcher on a baseball team says the path connecting home plate to first base has a length of 27.40 m. He is at rest relative to the path, so he measures the proper length, \( L_0 \).

\[ L_0 = 27.40 \text{ m} \]

The catcher observes a runner traveling from home plate to first base at a speed of 0.90 \( c \).

\[ v = 0.90 \times (3.0 \times 10^8) \]
\[ = 2.7 \times 10^8 \text{ m/s} \]

(a) What does the catcher say is the time it takes the runner to reach first base?

Time = Distance/Speed:
\[ = \frac{27.40 \text{ m}}{2.7 \times 10^8 \text{ m/s}} \]
\[ = 1.01 \times 10^{-7} \text{ s} \]

The event, which is the running, is taking place at the runner’s location. The catcher is moving relative to the event’s location (the runner’s location), so the catcher measures the dilated time:

\[ T = 1.01 \times 10^{-7} \text{ s} \]

(b) What does the runner say is the time it takes to reach first base?

The runner measures the proper time:

\[ T = \frac{T_0}{(1 - \beta^2)^{1/2}} \]
\[ T = \frac{1.01 \times 10^{-7}}{(1 - 0.90^2)^{1/2}} \]
\[ T_0 = 4.40 \times 10^{-8} \text{ s} \]

(c) What does the runner say is the length of the path?

The runner is moving relative to the path, so he measures the contracted length:

\[ L = L_0 \times (1 - \beta^2)^{1/2} \]
\[ = 27.40 \times (1 - 0.90^2)^{1/2} \]
\[ = 11.94 \text{ m} \]

(d) What does the runner say her speed is?

Runner Measures Distance = 11.94 m
Runner Measures Time = 4.40 \times 10^{-8} \text{ s}

Speed = Distance / Time
\[ = \frac{11.94 \text{ m}}{4.40 \times 10^{-8} \text{ s}} \]
\[ = 2.7 \times 10^8 \text{ m/s} \]

Note that this is the same speed the catcher measures. The two persons disagree about distance between home plate and first base, and they disagree about the time it takes the runner to run from home plate and first base, but they agree about the runner’s speed, i.e., they agree about the ratio of distance and time. The catcher says the distance is longer, and so is the time; the runner says the distance is shorter, and so is the time, but the ratios—the speeds—are the same.